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Equilibrium Uniqueness in Network Games with Strategic Substitutes*

Yann Rébillé[†] Lionel Richefort[‡]

Abstract

A local public goods game in weighted and directed networks is analyzed. Individual efforts are imperfect substitutes, players' preferences are heterogeneous and local externalities are non-uniform and asymmetric. Sufficient conditions under which the game admits a unique equilibrium are established in terms of the number of links between agents in the original network. It appears that these latter conditions for uniqueness are met if, and only if, the structure of relationships is *productive*. That is, a parallel can be established between network games with strategic substitutes and the input-output theory pioneered by Wassily Leontief.

Keywords: local public goods, Nash equilibrium, generalized degree, productive matrix, Leontief model.

JEL: A14, C72, H41.

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1 Introduction

In many economic phenomena, social relationships play a critical role. These interactions may have positive or negative effects on individual decisions, with various degrees of intensity, and can be represented through a network (or graph, or adjacency matrix). In this paper, we focus on network games with strategic substitutes. These games are particularly appropriate for understanding the voluntary provision of goods that are non-excludable in a geographic or social dimension. That is, we study the class of games in which a player is less willing to exert a positive action when he sees that his neighbors are doing so.

For this purpose, we extend the work of Bramoullé and Kranton (2007; henceforth BK) on local public goods by allowing networks to be directed and weighted, and by allowing players to have different preferences. This means that players' investments in public goods can be imperfect substitutes, and one player's investment may increase the payoffs of another player, but not vice versa. In contrast, the main focus of BK is on the case where players have identical preferences, and on undirected and unweighted networks.

While BK briefly discuss the weighted case[1] and the case where players have different preferences, this present paper establishes sufficient conditions in terms of the number of links between agents under which the game has a unique Nash equilibrium (Theorems 1 and 2). It appears that these uniqueness conditions are met if, and only if, the structure of relationships is *productive*. When this condition is met, the unique equilibrium profile can be characterized in terms of the Bonacich centrality vector (Theorem 3). Related works include Bloch and Zenginobuz (2007) who analyze the

[1]When efforts are imperfect substitutes, BK note that similar techniques can be used as in Ballester et al. (2006). See Section 6, p. 489. See also Bramoullé et al. (2011), p. 10.

uniqueness of noncooperative equilibria in a local public good game played by jurisdictions in presence of spillovers, and Ballester and Calvó-Armengol (2010) who study the uniqueness of equilibria in network games where best response functions are continuous and partially linear.

Following Banach's (1922) contraction mapping principle, our results generalize Bloch and Zenginobuz's (2007) sufficient condition under which a local public goods game admits a unique equilibrium if spillovers (or externalities) between players are small enough. Bloch and Zenginobuz's condition appears as a special case of Theorem 1 (Corollary 1). As another corollary, we show that best response functions contract if the highest *generalized degree* of the original network is sufficiently low (Corollary 2). Our results also extend BK's sufficient condition under which the game admits a unique equilibrium without free riders if the degree of substitutability between efforts is sufficiently low[2] (Corollary 3).

Using Gale's (1960) productive matrix terminology, this paper provides new interpretations, both in network and economic terms, of the condition established by Ballester and Calvò-Armengol (2010) under which a game with asymmetric substitutabilities admits a unique equilibrium if the spectral radius of the interaction matrix is low enough.[3] Our results yield three new insights. First, games with asymmetric substitutabilities admit a unique Nash equilibrium if there is no "star" in the network. Second, when this condition is met, the unique equilibrium profile can be characterized in terms of the Bonacich centrality measure. Third, a bridge can be established between network games with strategic substitutes and the Leontief production

[2]This result is established when networks are undirected, and when players have identical preferences. That is, the degree of imperfect substitutability between efforts is homogeneous and uniform across players.

[3]This condition is related to the Perron-Frobenius theory. See Proposition 1, p. 13. See also Theorem 1 in Ballester et al. (2006, p. 1408) or Theorem 1 in İlkiliç (2011, p. 112).

model, i.e., agents' network centrality and investment in public goods can be interpreted in terms of physical quantities.

The remainder of the paper is organized as follows. The next section presents the model. Section 3 establishes equilibrium uniqueness. In Section 4, we derive two equivalence results and in Section 5, we show how the model may be understood in terms of the statical model of the Leontief system. Section 6 concludes by discussing some extensions of the model. The main proofs are relegated to the appendix.

2 The model

Matrices are represented as bold upper case and column vectors as bold lower case. All vectors are column vectors unless explicitly written as transposed. The transpose of a matrix $\mathbf{M}$ is denoted $\mathbf{M}^T$. The transpose of a vector $\mathbf{v}$ is denoted $\mathbf{v}^T$. The coordinates of $\mathbf{v}$ are $(v_1, \ldots, v_N) \in \mathbb{R}^N$. Let $\mathbf{I}$ stands for the identity matrix and $\mathbf{1}$ for the vector of ones.

There are N agents embodied in a social network. The basic representation of the network is given by its $N \times N$ weighted adjacency matrix $\mathbf{\Lambda} = [\lambda_{kl}]$ in which the entry $\lambda_{kl} > 0$ if there is a link from vertex k to vertex l, otherwise $\lambda_{kl} = 0$. Since $\mathbf{\Lambda}$ is directed, the neighbors of an agent are divided into two sets: her predecessors and her successors. We may refer to $\mathbf{\Lambda}$ as either the (weighted) interaction matrix, or the network.

We shall consider from now on quasi-linear preferences. The utility of an agent l is given by

$$U_l(\mathbf{\Lambda}, \mathbf{e}) = v_l e_l + w_l(e_l + \sum_{k:k \neq l} \lambda_{kl} e_k),$$

where e_l denotes own effort and $\sum_{k:k\neq l}\lambda_{kl}e_k$ reflects predecessors' efforts. The boundary conditions are given by $w_l'(0) > -v_l > w_l'(\infty)$. That is, any agent will provide some effort and preferences are single-peaked w.r.t. own effort. Therefore, efforts are imperfect substitutes and allow a trade-off between own effort and predecessors' efforts. We also assume that $w_l'' < 0$. This assumption reflects the convexity of preferences. Utility is cardinal and admits an interpretation in terms of benefits and costs. To give concrete examples, we provide two economic applications of this framework.

Application 1. This is BK's model (extended). Given a network between agents, this model studies the incentives to provide a local public good which is non-excludable along the links. Consider, for instance, a set of agents who can plant an attractive garden in front of their house. The agents decide on the amount of effort to produce the good (the attractive garden) and given the effort, a strictly concave production function determines how much good is produced by the agent. Each agent incurs the cost from her individual effort. Since planting an attractive garden may provide benefits to others living in the area (even financial benefits) agents benefit both from their own contribution and from the contributions of their direct neighbors. An agent l's payoff function is written as follows:

$$U_l(\mathbf{\Lambda}, \mathbf{e}) = b_l(e_l + \sum_{k:k\neq l}\lambda_{kl}e_k) - c_l e_l$$

where $b_l(.)$ is an increasing twice differentiable strictly concave benefit function and c_l are individual marginal costs. It is assumed that $c_l > 0$ and the boundary conditions are $b_l'(0) > c_l > b_l'(\infty)$.

Application 2. This is the "dual" model of BK's model. Consider a set of linked agents that exploit a common property resource, say for instance a fishery. The agents decide on their fishing activity and given this quantity, a strictly convex cost function determines how many fish is harvested by the agent. Each agent benefits from her fishing activity. Since one agent's fishing activity may decrease the local stock of fish, agents incur the cost of their own fishing activity and from the fishing activity of their direct neighbors (those fishing from the same source). An agent l's payoff function is written as follows:

$$U_l(\mathbf{\Lambda}, \mathbf{e}) = p_l e_l - q_l(e_l + \sum_{k:k \neq l} \lambda_{kl} e_k)$$

where $q_l(.)$ is an increasing twice differentiable strictly convex cost function and p_l denote individual marginal benefits. It is assumed that $p_l > 0$ and the boundary conditions are $q_l'(0) < p_l < q_l'(\infty)$.

We specify a game in which players simultaneously choose their effort level by playing their best response to the effort level played by their predecessors. Given an effort profile $\mathbf{e}$, each agent l earns payoffs $U_l(\mathbf{\Lambda}, \mathbf{e})$. At this point, we think it useful to give a reminder of the definition of a Nash equilibrium.

Definition 1. Let $\mathbf{\Lambda}$ be a network. A profile $\hat{\boldsymbol{e}} \in \mathbb{R}_+^N$ is a *Nash equilibrium* if and only if, for all agent l, for all $\hat{e}_l' \geq 0$,

$$U_l(\mathbf{\Lambda}, \hat{e}_l, \hat{\boldsymbol{e}}_{-\boldsymbol{l}}) \geq U_l(\mathbf{\Lambda}, \hat{e}_l', \hat{\boldsymbol{e}}_{-\boldsymbol{l}}).$$

For each agent l, we note e_l^* her effort level such that $w_l'(e_l^*) = -v_l$. We note $\mathbf{e}^*$ the vector of individual peaks, and $G(\mathbf{\Lambda}, \mathbf{e}^*)$ the game played by the agents. The vector $\mathbf{e}^*$ keeps track of the marginal cost and marginal benefit of each agent. We may refer to e_l^* as either the weight or the peak assigned to agent l. Given a game $G(\mathbf{\Lambda}, \mathbf{e}^*)$, agents want to exert some effort as long as their peak is higher than a weighted amount of effort exerted by their predecessors.

Property 1. *Let $G(\mathbf{\Lambda}, \mathbf{e}^*)$ be a network game. A profile $\hat{\mathbf{e}} \in \mathbb{R}_+^N$ is a* Nash equilibrium *if and only if, for all l,*

$$\hat{e}_l = \max\{0, e_l^* - \hat{\mathbf{e}}^T \mathbf{\Lambda}_{.l}\}.$$

Since we know this game always admits a (pure-strategy) Nash equilibrium[4], we shall focus on the uniqueness of Nash equilibria.[5]

3 Generalized out-degree and uniqueness

Before turning to the equilibrium uniqueness analysis of the game $G(\mathbf{\Lambda}, \mathbf{e}^*)$, we introduce a centrality measure that generalizes the concept of out-degree and in-degree. In this section, we focus on the out-degree of a node. In any directed network where both links and nodes are weighted, the out-degree of an agent should measure her "ability to give". This measure may depend both on the interaction matrix and on the vector of weights assigned to agents.[6] In directed and weighted networks, the out-degree is generalized as follows.

Definition 2. Let $\mathbf{\Lambda}$ be a network and $\mathbf{y} >> 0$ a vector of weights assigned

[4]The set of individual best responses $\mathbf{f} = (f_1, ..., f_N)$ defines a continuous mapping from $\prod_l [0, e_l^*]$, a convex compact subset of a Euclidean space, into itself. By Brouwer's fixed point theorem, $\mathbf{f}$ has a fixed point, i.e., there is a point $\mathbf{e}$ such that $\mathbf{f}(\mathbf{e}) = \mathbf{e}$, and the existence of a pure-strategy Nash equilibrium is guaranteed.

[5]One may note that if an effort profile is a Nash equilibrium of game $G(\mathbf{\Lambda}, \mathbf{e}^*)$, then the effort profile in which the free riders have been deleted is a Nash equilibrium without free riders of the corresponding subgame. Therefore, one may notice that to any Nash equilibrium profile $\hat{\mathbf{e}}$, we may associate a *type*, i.e., its set of free riders. Since this set is empty whenever $\hat{\mathbf{e}}$ is a Nash equilibrium without free riders, there are $2^N - 1$ types of Nash equilibria. Moreover, when there are multiple Nash equilibria without free riders, their set is always a continuum. Consequently, either the number of Nash equilibria is infinite or it is less than $2^N - 1$.

[6]Note that there are two kinds of weights in the model, i) weight (intensity) of a link and ii) weight (peak) of a player.

to nodes. For all i, the *generalized out-degree* of agent i is given by:

$$\delta_i^+(\mathbf{\Lambda}, \mathbf{y}) = \sum_{l:l \neq i} \lambda_{il} \frac{y_i}{y_l}.$$

Let us denote by $\Lambda_{\mathbf{y}}^+$ the greatest generalized out-degree of the network, i.e.,

$$\Lambda_{\mathbf{y}}^+ = \max_i \delta_i^+(\mathbf{\Lambda}, \mathbf{y}).$$

Following this definition, we say a directed network is *generalized* when it has both weights assigned to links and nodes. If weights assigned to nodes are uniform and homogeneous, the generalized out-degree of an agent simply becomes her weighted out-degree, i.e., for all i, $\delta_i^+(\mathbf{\Lambda}, \mathbf{y}) = \sum_{l:l \neq i} \lambda_{il}$. If links are unweighted, i.e., for all $i \neq l$, $\lambda_{il} \in \{0, 1\}$, the generalized out-degree of an agent becomes $\delta_i^+(\mathbf{\Lambda}, \mathbf{y}) = \sum_{l:\lambda_{il}=1} \frac{y_i}{y_l}$.

In a generalized network $(\mathbf{\Lambda}, \mathbf{y})$, any agent whose generalized out-degree is less than 1 will be called a *sink*. Given a network $\mathbf{\Lambda}$ between agents, we show that if there exists a vector of weights $\mathbf{y} >> 0$ assigned to nodes so that any agent is a sink, i.e., there is no agent who could potentially give "too much" to her successors, the game $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium, for any vector of peaks assigned to players.

Theorem 1. *Let $\mathbf{\Lambda}$ be a network. If there exists $\mathbf{y} >> 0$ such that $\Lambda_{\mathbf{y}}^+ < 1$, then $\forall \mathbf{e}^*$, $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium.*

Given a game $G(\mathbf{\Lambda}, \mathbf{e}^*)$, equilibrium uniqueness is guaranteed whatever the marginal benefit and marginal cost of effort of each agent if there is one network $(\mathbf{\Lambda}, \mathbf{y})$ in which the highest generalized out-degree is sufficiently low. For any network between agents, it suffices that there exists one arbitrary vector $\mathbf{y}$, that is not necessarily the vector of peaks $\mathbf{e}^*$, such that all agents have a sufficiently low generalized out-degree, to establish uniqueness in the set of Nash equilibria of game $G(\mathbf{\Lambda}, \mathbf{e}^*)$. This sufficient condition concerns the structure of the interation matrix $\mathbf{\Lambda}$, i.e., the number, the intensity and the direction of relationships. We illustrate this result through the following example.

Example 1. Consider a game $G(\mathbf{\Lambda}, \mathbf{e}^*)$ with two players. Let

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & \lambda' \\ \lambda & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e}^* = \begin{pmatrix} 1 \\ e_2^* \end{pmatrix}.$$

The corresponding graph is depicted in the following figure.

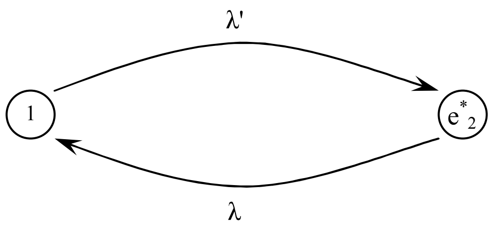

Figure 1: Directed circle with two agents

For ease of exposition, but w.l.o.g., we assume that $\lambda \in]0,1]$ and $\lambda' \in]0,1]$.

Since $\Lambda_{\mathbf{y}}^+$ is homogeneous of degree 0, let us take $\mathbf{y} = (1, y_2)$. Then, we may derive the generalized out-degree of agents in network $(\mathbf{\Lambda}, \mathbf{y})$. We obtain $\delta_1^+(\mathbf{\Lambda}, \mathbf{y}) = \lambda' y_2$ and $\delta_2^+(\mathbf{\Lambda}, \mathbf{y}) = \frac{\lambda}{y_2}$. It appears that $\max\{\delta_1^+, \delta_2^+\} < 1$ if and only if $\lambda' y_2 < 1$ and $\frac{\lambda}{y_2} < 1$. That is, $\Lambda_{\mathbf{y}}^+ < 1$ if and only if $\lambda < \frac{1}{\lambda'}$.

It follows directly that if $\lambda\lambda' < 1$, the game $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium, whatever $e_2^* > 0$.

Note that if $\lambda\lambda' = 1$ and $e_2^* = 1$, the game admits a continuum of Nash equilibria. If $\lambda\lambda' = 1$ and $e_2^* \neq 1$, however, the game admits a unique Nash equilibrium. This point illustrates the fact that the condition provided by Theorem 1 is only sufficient.

As a first corollary of Theorem 1, we derive an extension of Bloch and Zenginobuz's (2007) sufficient condition.[7] Given a network between agents, the game $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium if each agent is a sink in the generalized network $(\mathbf{\Lambda}, \mathbf{1})$. This condition is met if and only if the highest weighted out-degree of network $\mathbf{\Lambda}$ is sufficiently low. That is, when individual preferences are homogeneous, each agent should have few successors and/or low weights assigned to arcs starting from her. In that case, $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique equilibrium, whatever the marginal cost and marginal benefit of effort of each agent.

Corollary 1. *Let $\mathbf{\Lambda}$ be a network. If for all i, $\sum_{l:l\neq i} \lambda_{il} < 1$, then $\forall \mathbf{e}^*$, $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium.*

Proof. $\Lambda^+_{\mathbf{y}} = \max_i \sum_{l:l\neq i} \lambda_{il}$ with $\mathbf{y} = \mathbf{1}$. □

A second corollary emerges naturally. Given a network $\mathbf{\Lambda}$ between agents and a vector of individual peaks $\mathbf{e}^*$, the game $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium if the highest generalized out-degree of the network $(\mathbf{\Lambda}, \mathbf{e}^*)$ is sufficiently low. This means that equilibrium uniqueness is guaranteed for any set of individual preferences if each agent is a sink in the original generalized network.

Corollary 2. *Let $G(\mathbf{\Lambda}, \mathbf{e}^*)$ be a network game. If $\Lambda^+_{\mathbf{e}^*} < 1$, there exists a unique Nash equilibrium. Moreover, any network game $G(\mathbf{\Lambda}, \mathbf{a})$ with $\mathbf{a} >> 0$ and $\mathbf{a} \neq \mathbf{e}^*$ admits a unique Nash equilibrium.*

Proof. Consider $\Lambda^+_{\mathbf{y}}$ with $\mathbf{y} = \mathbf{e}^*$. □

[7] See Proposition 7 p. 211.

These latter results are less general than Theorem 1. Consider a game $G(\mathbf{\Lambda}, \mathbf{e}^*)$ such that $\Lambda^+_{\mathbf{e}^*} > 1$ and $\Lambda^+_{\mathbf{1}} > 1$. If there exists one positive vector $\mathbf{y}$, different from $\mathbf{1}$ and $\mathbf{e}^*$, such that $\Lambda^+_{\mathbf{y}} < 1$, then $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium.

4 Equivalence results

4.1 Generalized in-degree and uniqueness

We derive two results that are equivalent to Theorem 1. First, we focus on the in-degree of a node. In generalized networks, its definition is slightly different than those of the out-degree. The generalized in-degree of an agent should reflect her "ability to receive", when preferences are heterogeneous. Given a network between agents, the generalized in-degree of an agent decreases with her own peak and increases with her predecessors' peaks. This centrality measure is defined as follows.

Definition 3. Let $\mathbf{\Lambda}$ be a network and $\mathbf{y} >> 0$ a vector of weights assigned to nodes. For all i, the *generalized in-degree* of agent i is given by:

$$\delta_i^-(\mathbf{\Lambda}, \mathbf{y}) = \sum_{k:k\neq i} \lambda_{ki} \frac{y_k}{y_i}.$$

Let us denote by $\Lambda^-_{\mathbf{y}}$ the greatest generalized out-degree of the network, i.e.,

$$\Lambda^-_{\mathbf{y}} = \max_i \delta_i^-(\mathbf{\Lambda}, \mathbf{y}).$$

In a generalized network $(\mathbf{\Lambda}, \mathbf{y})$, any agent whose generalized in-degree is less than 1 will be called a *source*. Given a network $\mathbf{\Lambda}$ between agents, we show that the game $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium for any vector of peaks assigned to players if there exists a vector of weights $\mathbf{y} >> 0$ assigned to nodes such that each agent is a source in the generalized network $(\mathbf{\Lambda}, \mathbf{y})$, i.e., there exists one generalized network in which no agent could potentially receive "too much" from her predecessors.

Theorem 2. *Let $\mathbf{\Lambda}$ be a network. If there exists $\mathbf{y} >> 0$ such that $\Lambda^-_{\mathbf{y}} < 1$, then $\forall \mathbf{e}^*$, $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium.*

When the network is undirected, the interaction matrix $\mathbf{\Lambda}$ is symmetric and we have $\Lambda^+_{\mathbf{1}} = \Lambda^-_{\mathbf{1}}$. When the network is directed, the highest generalized out-degree of network $(\mathbf{\Lambda}, \mathbf{1})$ is equal to the highest generalized in-degree of network $(\mathbf{\Lambda}^T, \mathbf{1})$, i.e., $\Lambda^+_{\mathbf{1}} = (\Lambda^T)^-_{\mathbf{1}}$. This property can be extended to any generalized network $(\mathbf{\Lambda}, \mathbf{y})$.

Remark 1. For any $\mathbf{y} >> 0$, each agent's generalized out-degree in network $(\mathbf{\Lambda}, \mathbf{y})$ is equal to her generalized in-degree in network $(\mathbf{\Lambda}^T, \mathbf{1}/\mathbf{y})$, i.e.,

$$\delta^+(\mathbf{\Lambda}, \mathbf{y}) = \delta^-(\mathbf{\Lambda}^T, \mathbf{1}/\mathbf{y}),$$

where $\mathbf{1}/\mathbf{y} = (\frac{1}{y_1}, \ldots, \frac{1}{y_N})$. It follows that the highest generalized out-degree of network $(\mathbf{\Lambda}, \mathbf{y})$ is equal to the highest generalized in-degree of network $(\mathbf{\Lambda}^T, \mathbf{1}/\mathbf{y})$, i.e.,

$$\Lambda^+_{\mathbf{y}} = (\Lambda^T)^-_{\mathbf{1}/\mathbf{y}}.$$

Therefore, there exists a source in network $(\mathbf{\Lambda}, \mathbf{y})$ if and only if there exists a sink in network $(\mathbf{\Lambda}^T, \mathbf{1}/\mathbf{y})$.

Consider a game $G(\mathbf{\Lambda}, \mathbf{e}^*)$ such that $\Lambda^-_{\mathbf{e}^*} < 1$. This game admits a unique Nash equilibrium, whatever the individual preferences. In that case, no player may receive enough from his predecessors to exert no effort. That is, if the highest generalized in-degree of the original network $(\mathbf{\Lambda}, \mathbf{e}^*)$ is sufficiently low, the game $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium which is without free riders. The following property extends BK's result to the case of imperfect substitutability of efforts in directed networks.

Property 2. $\Lambda^-_{\mathbf{e}^*} < 1 \iff \mathbf{e}^{*T}(\mathbf{I} - \mathbf{\Lambda}) >> 0 \implies \min_i \hat{e}_i > 0.$

BK's result may also be generalized as a corollary of Theorem 2. Given a network $\mathbf{\Lambda}$ between agents, suppose that the highest generalized in-degree of a network $(\mathbf{\Lambda}, \mathbf{1})$, i.e., the highest weighted in-degree of network $\mathbf{\Lambda}$, is sufficiently low. Then, $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique equilibrium if individual preferences $\mathbf{e}^*$ are homogeneous, whatever the level of $\mathbf{e}^*$. That is, there exists a unique Nash equilibrium for any (homogeneous) marginal cost and marginal benefit of effort.

Corollary 3. *Let $\mathbf{\Lambda}$ be a network with $\Lambda^-_{\mathbf{1}} < 1$. If the peaks are homogeneous, i.e., $\mathbf{e}^* = \varepsilon\mathbf{1}$ with $\varepsilon > 0$, then $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium (which is without free riders).*

Proof.

$$\Lambda^-_{\mathbf{1}} < 1 \iff \mathbf{1}^T(\mathbf{I} - \mathbf{\Lambda}) >> 0 \iff \mathbf{e}^{*T}(\mathbf{I} - \mathbf{\Lambda}) >> 0.$$

□

4.2 Productive matrix and uniqueness

This subsection establishes equivalence between the uniqueness condition related to individual out-degree (Theorem 1) and those related to individual in-degree (Theorem 2). The bridge is made using the concept of *productive matrix* first introduced by David Gale (1960). A linear model of production is called productive if, for each set of prices, it produces a unique quantity of output. In that case, we may say that the production matrix itself is productive. A productive matrix is defined as follows.

Definition 4. Let $\mathbf{\Lambda} \geq 0$ be a matrix. $\mathbf{\Lambda}$ is *productive* if there exists[8] $\mathbf{y} >> 0$ such that $\mathbf{y} - \mathbf{\Lambda y} >> 0$, that is

$$(\mathbf{I} - \mathbf{\Lambda})\mathbf{y} >> 0 \iff \mathbf{y}^T(\mathbf{I} - \mathbf{\Lambda}^T) >> 0.$$

It appears that $\exists \mathbf{y} >> 0$ such that $\Lambda_{\mathbf{y}}^{+} < 1$ if and only if $(\mathbf{I} - \mathbf{\Lambda})\mathbf{1}/\mathbf{y} >> 0$. Since $\mathbf{1}/\mathbf{y} >> 0$, this latter condition coincides with the definition of productive matrix. The following property establishes equivalence between the sufficient condition related to out-degree and the fact that the interaction matrix is a productive matrix.

Property 3. *Let* $\mathbf{\Lambda}$ *be a network. Then, there exists* $\mathbf{y} >> 0$ *such that* $\Lambda_{\mathbf{y}}^{+} < 1$ *if and only if* $\mathbf{\Lambda}$ *is a productive matrix.*

Given a game $G(\mathbf{\Lambda}, \mathbf{e}^*)$, if the structure of relationships is productive, $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique equilibrium, whatever $\mathbf{e}^*$. We illustrate this result through the following extension of Example 1.

Example 2 (Example 1 continued)**.** Let us check under which conditions the interaction matrix of the game $G(\mathbf{\Lambda}, \mathbf{e}^*)$ is productive.[9] To achieve this, we may calculate the inverse of the matrix $(\mathbf{I} - \mathbf{\Lambda})$, i.e.,

$$(\mathbf{I} - \mathbf{\Lambda})^{-1} = \frac{1}{1 - \lambda\lambda'} \begin{pmatrix} 1 & \lambda' \\ \lambda & 1 \end{pmatrix}.$$

[8] It suffices that $\mathbf{y} \geq 0$ such that $\mathbf{y} - \mathbf{\Lambda y} >> 0$ since $\mathbf{\Lambda} \geq 0$ and $\mathbf{y} \geq 0$ implies that $\mathbf{\Lambda y} \geq 0$; and $\mathbf{y} >> \mathbf{\Lambda y} \geq 0$ implies that $\mathbf{y} >> 0$.

[9] We remind that a nonnegative square matrix $\mathbf{\Lambda}$ is productive if and only if $(\mathbf{I} - \mathbf{\Lambda})^{-1}$ exists and is nonnegative.

One notes that $(\mathbf{I}-\mathbf{\Lambda})^{-1}$ exists and is nonnegative if and only if $\lambda\lambda' < 1$. In that case, the interaction matrix $\mathbf{\Lambda}$ is productive, and equilibrium uniqueness is guaranteed. Then, we obtain the same uniqueness condition than those related to the highest generalized out-degree.

One important feature of $G(\mathbf{\Lambda}, \mathbf{e}^*)$ is that this game is "invertible". Since $\Lambda^+_{\mathbf{y}} = (\Lambda^T)^-_{\mathbf{1/y}}$, games $G(\mathbf{\Lambda}, \mathbf{e}^*)$ and $G(\mathbf{\Lambda}^T, \mathbf{e}^*)$ exhibit common properties. It appears that $\exists \mathbf{y} >> 0$ such that $\Lambda^-_{\mathbf{y}} < 1$ if and only if $(\mathbf{I}-\mathbf{\Lambda}^T)\mathbf{y} >> 0$. Since $\mathbf{y} >> 0$, this latter condition coincides with the definition of productive matrix. The following property establishes equivalence between the sufficient condition related to in-degree and the fact that the transpose of the interaction matrix is a productive matrix.

Property 4. *Let $\mathbf{\Lambda}$ be a network. Then, there exists $\mathbf{y} >> 0$ such that $\Lambda^-_{\mathbf{y}} < 1$ if and only if $\mathbf{\Lambda}^T$ is a productive matrix.*

Since a matrix is productive if and only if its transpose is productive, we obtain the following equivalence:

$$\exists \mathbf{y} >> 0 \text{ s.t. } \Lambda^+_{\mathbf{y}} < 1 \iff \mathbf{\Lambda} \text{ is productive } \iff \exists \mathbf{z} >> 0 \text{ s.t. } \Lambda^-_{\mathbf{z}} < 1.$$

Moreover, a nonnegative square matrix is productive if, and only if, its spectral radius is smaller than 1.[10] Consequently, Theorems 1 and 2 provide equivalent results to the spectral condition established by Ballester and Calvó-Armengol (2010). Our results add new insights. Let us call a *star* any agent who is neither a sink nor a source in network $\mathbf{\Lambda}, \mathbf{y})$, for all $\mathbf{y} >> 0$. We know that Ballester and Calvó-Armengol's condition is met if, and only if, there is no star in the network of relationships. This appears if, and only

[10]See, e.g., Ivanov (2001), Theorem 2.4 (p. 13).

if, the network of relationships is productive. That is, BK's model may be understood in terms of the Leontief production model.

5 Economic interpretation

5.1 The fundamental relation of BK's model

Before turning to the economic interpretation of BK's model, we provide a characterization of the equilibrium effort profile in terms of the centrality of a player in the network. We obtain it by using the *weighted Bonacich centrality* measure, which has proven to be very useful in games with complementarities (Ballester et al., 2006; Ballester and Calvó-Armengol, 2010). This measure is a weighted version of the network centrality measure introduced by Philip Bonacich (1987), and is defined as follows.

Definition 5. Let $\mathbf{\Lambda}$ be a (weighted) network and $\mathbf{u} >> 0$ a vector of weights assigned to nodes. The vector of *weighted Bonacich centrality* (of parameter 1) is given by:

$$\mathbf{b}_{\mathbf{u}}(\mathbf{\Lambda}) = (\mathbf{I} - \mathbf{\Lambda})^{-1}\mathbf{u},$$

provided that $\mathbf{\Lambda}$ is a productive matrix.[11]

In the following analysis, we will assume that $\mathbf{\Lambda}$ is a productive matrix. When this condition is met, we know that $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium. The next property characterizes the unique Nash equilibrium profile in terms of the weighted Bonacich centrality of a network related to the interaction matrix. We call this characterization the *fundamental relation of BK's model*.

[11]Since $\mathbf{\Lambda}$ is productive, the matrix $(\mathbf{I} - \mathbf{\Lambda})^{-1}$ is well-defined and nonnegative.

Theorem 3. *Let $G(\mathbf{\Lambda}, \mathbf{e}^*)$ be a network game with $\mathbf{\Lambda}$ a productive matrix and $\hat{\mathbf{e}}$ the (unique) Nash equilibrium. If $\hat{\mathbf{e}}$ is without free riders, i.e., $\hat{\mathbf{e}} >> 0$, then*

$$\hat{\mathbf{e}} = (\mathbf{I} - \mathbf{\Lambda}^T)\mathbf{b}_{\mathbf{e}^*}(\mathbf{\Omega})$$

where $\mathbf{\Omega} = (\mathbf{\Lambda}^T)^2$.

The following remark shows that this characterization is valid for any unique Nash equilibrium, i.e., not only in the event of no free riders, provided that $\mathbf{\Lambda}$ is a productive matrix.

Remark 2. Let $\hat{\mathbf{e}}$ be a (unique) Nash equilibrium with free riders and $L = \{l : \hat{e}_l > 0\}$ the set of active agents. Consider the subgame $G(\mathbf{\Lambda}_L, \mathbf{e}^*{}_L)$ where $\mathbf{\Lambda}_L$ denotes the new network obtained after deletion of all the free riders. Then,

$$\hat{\mathbf{e}}_L = (\mathbf{I} - \mathbf{\Lambda}_L{}^T)\mathbf{b}_{\mathbf{e}^*{}_L}(\mathbf{\Omega}_L)$$

where $\mathbf{\Omega}_L = (\mathbf{\Lambda}_L{}^T)^2$.

Since we allow networks to be directed, this result generalizes the result established by Ballester and Calvó-Armengol (2010).[12] When networks are directed, the characterization of the unique Nash equilibrium in terms of the Bonacich centrality measure involves the *transpose* of the interaction matrix. This makes sense, since the vector $\mathbf{\Lambda}^T\mathbf{b}_{\mathbf{e}^*}(\mathbf{\Omega})$ reflects what players "benefit" from their predecessors. This point is ignored when networks are undirected. We illustrate this result through the following example.

Example 3 (Example 1 continued)**.** One may use the fundamental relation of BK's model to calculate the Nash equilibrium profile of the game $G(\mathbf{\Lambda}, \mathbf{e}^*)$ presented in Example 1. In this simple game, we have $\mathbf{\Lambda}^2 = (\mathbf{\Lambda}^2)^T = (\lambda\lambda')\mathbf{I}$.

[12]See Proposition 2 (p. 403).

It follows that $\mathbf{\Omega} = (\lambda\lambda')\mathbf{I}$, and therefore,

$$\mathbf{b}_{\mathbf{e}^*}(\mathbf{\Omega}) = \frac{1}{1-\lambda\lambda'}\mathbf{e}^*.$$

Then, we obtain the unique Nash equilibrium profile given by

$$\hat{\mathbf{e}} = \frac{1}{1-\lambda\lambda'}\begin{pmatrix} 1-\lambda e_2^* \\ e_2^* - \lambda' \end{pmatrix}.$$

This characterization is valid when the equilibrium profile is interior, that is, if $e_2^* \in]\lambda', \frac{1}{\lambda}[$. Otherwise, the unique Nash equilibrium profile is corner. If $e_2^* \leq \lambda'$, player 2 does not provide any effort and player 1 plays his peak (which is his weighted Bonacich centrality in the new network obtained after deletion of player 2). In that case, the equilibrium profile is given by $\{(1,0)\}$. If $e_2^* \geq \frac{1}{\lambda}$, player 1 does not provide any effort and player 2 plays his peak (which is his weighted Bonacich centrality in the new network obtained after deletion of player 1). In that case, the equilibrium profile is given by $\{(0, e_2^*)\}$.

As in Ballester and Calvó-Armengol (2010), our characterization of the unique Nash equilibrium profile is not in terms of the Bonacich centrality of a player in the original network, but in terms of the Bonacich centrality in *another* network. The interaction matrix of this new network is the square of the original interaction matrix. According to Ballester and Calvó-Armengol, this reflects the idea that games with substitutabilities may be understood as games with hidden complementarities, since "local network substitutabilities induce complementarities for players who are two-links-away from each other in the network" (p. 403). We now provide an economic interpretation of this result. We begin by showing the similarity between the fundamental relation of BK's model and the balance equations for total output of the Leontief production model.

5.2 Interpretation in terms of physical quantities

In matrix form, the statical model of the Leontief system (henceforth the Leontief model) establishes a relation between a vector of producible outputs $\mathbf{x}$, a vector of final consumption of each of the produced goods $\mathbf{c}$ and a vector of intermediate inputs.[13] This latter vector makes use of the special fixed-coefficient assumption, i.e., each intermediate input is required in fixed proportion to any output. These requirements are specified by the means of a square technological matrix $\mathbf{A}$, and the vector $\mathbf{Ax}$ represents the vector of intermediate inputs. The balance equations for total output is written as follows:

$$\mathbf{x} = \mathbf{Ax} + \mathbf{c}.$$

Since $\mathbf{x} = \mathbf{Ax} + \mathbf{c} \iff \mathbf{c} = (\mathbf{I} - \mathbf{A})\mathbf{x}$, the proximity between the balance equations for total output of the Leontief model and the fundamental relation of BK's model, i.e., $\hat{\mathbf{e}} = (\mathbf{I} - \mathbf{\Lambda}^T)\mathbf{b}_{\mathbf{e}^*}(\mathbf{\Omega})$, is immediate. The equilibrium level of public good provision $\hat{\mathbf{e}}$ in BK's model may be assimilated to the vector of final consumption in the Leontief model. Furthermore, the vector of Bonacich centrality $\mathbf{b}_{\mathbf{e}^*}(\mathbf{\Omega})$ may be understood as the vector of producible outputs, and the vector $\mathbf{\Lambda}^T\mathbf{b}_{\mathbf{e}^*}(\mathbf{\Omega})$ as the vector of intermediate inputs. The technological matrix of the Leontief model is the transpose of the (fixed) interaction matrix in BK's model. It follows that the equilibrium level of public good provision $\hat{\mathbf{e}}$ in BK's model denotes the *net production* of effort.

Many authors have focused on the existence and uniqueness of a nonnegative solution to the Leontief model. We remind a central theorem established

[13] See, e.g., Dorfman et al. (1958, chapters 9 and 10), for a detailed presentation of the Leontief production model.

by Dorfman et al. (1958) and Gale (1960).

Theorem (Dorfman et al., 1958; Gale, 1960)**.** *If* $\mathbf{A}$ *is productive then for any nonnegative (final consumption) vector* $\mathbf{c}$ *there exists a unique nonnegative (producible output) vector* $\mathbf{x}$ *such that*

$$\mathbf{c} = (\mathbf{I} - \mathbf{A})\mathbf{x}.$$

In light of the parallel established between BK's model and the Leontief model, the following important insight appears. If the structure of relationships $\mathbf{\Lambda}$ is productive, for any (unique) equilibrium profile of public good provision $\hat{\mathbf{e}}$ seen as a final consumption (or net production) vector there exists a unique producible output vector which turns out to be the nonnegative weighted Bonacich centrality vector $\mathbf{b}_{\mathbf{e}^*}(\mathbf{\Omega})$. That is, our network game-theoretic analysis provides a new interpretation of the solution to the Leontief production model. Moreover, this solution is fully characterized in terms of both the weights assigned to players and the structure (number, intensity, direction) of relationships between players.

6 Conclusion

This paper analyzes a local public good games in which heterogeneous players are embodied in a directed and weighted network. Our main results exhibit sufficient conditions in terms of the number of links between agents in the original network to guarantee a unique level of public good provision. That is, for any individual marginal benefit and marginal cost of effort, network games with strategic substitutes admit a unique equilibrium if there is no *star* in the network. The same reasoning applies to the highest generalized in-degree.

It appears that these latter conditions for uniqueness are both equivalent to the fact that the structure of relationships is *productive*. When this condition is met, Nash equilibria can be implemented through the Bonacich centrality measure. We show how this result can be interpreted in economic terms within the Leontief production model.

A useful direction for further reasearch would be to investigate more general payoff functions, in particular the case of separable additive utility functions. This will lead us to study nonlinear best response functions.

Appendix

Proof of Property 1. Without externalities, each agent l maximizes her utility with respect to her own level of effort:

$$\underset{e_l}{\text{Max}}\ v_l e_l + w_l(e_l)$$
$$\text{s.t. } e_l \geq 0.$$

Let e_l^* denotes the individual effort level such that $w_l'(e_l^*) = -v_l$. Thus, e_l^* corresponds to agent l's peak, which is positive because $w_l'(0) > -v_l$.

With externalities, the program of agent l is:

$$\underset{e_l}{\text{Max}}\ v_l e_l + w_l(e_l + \sum_{k:k\neq l} \lambda_{kl} e_k)$$
$$\text{s.t. } e_l \geq 0.$$

The Kuhn-Tucker's conditons are

$$w_l'(e_l + \sum_{k:k\neq l} \lambda_{kl} e_k) + v_l + \mu_l = 0$$

with $\mu_l e_l = 0$ and $\mu_l \geq 0$, where μ_l is a Lagrange multiplier. Thus, the best responses are

$$\hat{e}_l = \begin{cases} e_l^* - \sum\limits_{k:k\neq l} \lambda_{kl} e_k, & \text{if } w_l'(\sum\limits_{k:k\neq l} \lambda_{kl} e_k) \geq -v_l. \\ 0, & \text{otherwise.} \end{cases}$$

Let $K_{-l} = \sum_{k:k\neq l} \lambda_{kl} e_k$. We see that if $w_l'(K_{-l}) \leq -v_l$, then $K_{-l} \geq (w_l')^{-1}(-v_l) = e_l^*$. Consequently, agents' preferences are single-peaked w.r.t. effort. $\square$

We introduce some useful notations for the following proofs.

- Let $\mathbf{e} \in \mathbb{R}_+^N$. The N_1 norm is given by

$$\|\mathbf{e}\|_1 = \sum_{l=1}^{N} |e_l|$$

  and the N_∞ norm is given by

$$\|\mathbf{e}\|_\infty = \max_{l=1,\ldots,N} |e_l|.$$

- Let $\mathbf{e} \geq 0$ and $\mathbf{y} >> 0$. Put,

$$\mathbf{e}/\mathbf{y} = (e_k/y_k)_k.$$

- We can define the following norms. Let $\mathbf{e} \in \mathbb{R}_+^N$,

$$\|\mathbf{e}\|_{1,\mathbf{y}} = \|\mathbf{e}/\mathbf{y}\|_1.$$

  and

$$\|\mathbf{e}\|_{\infty,\mathbf{y}} = \|\mathbf{e}/\mathbf{y}\|_{\infty}.$$

- Let $x^+ = \max\{x, 0\}$ denote the *positive part* of x. We remind that the function $(.)^+ = \max\{., 0\}$ is 1-lipschitz. For all $x, y \in \mathbb{R}$,

$$|x^+ - y^+| \leq |x - y|.$$

Proof of Theorem 1. For $\mathbf{e} \in \prod_l [0, e_l^*]$, the best response of each player l is given by

$$f_l(\mathbf{e}) = (e_l^* - \mathbf{e}^T \mathbf{\Lambda}_{.l})^+.$$

Since $\mathbf{\Lambda} \geq 0$, $f_l(\mathbf{e}) \leq e_l^*$. Hence we can introduce the best response mapping:

$$f : \prod_l [0, e_l^*] \longrightarrow \prod_l [0, e_l^*] : \mathbf{e} \mapsto f(\mathbf{e}) = (f_l(\mathbf{e}))_l.$$

We shall establish that this mapping is a contraction with respect to the specific norm, $\| \, . \, \|_{1,\mathbf{y}}$. Since $\|.\|_{1,\mathbf{y}}$ and $\|.\|_1$ are equivalent norms, $\prod_l [0, e_l^*]$ remains a complete space. Let $\mathbf{e}, \mathbf{e}' \in \prod_l [0, e_l^*]$. We have:

$$\begin{aligned}
\|f(\mathbf{e}) - f(\mathbf{e}')\|_{1,\mathbf{y}} &= \|(f(\mathbf{e}) - f(\mathbf{e}'))/\mathbf{y}\|_1 \\
&= \|((f_l(\mathbf{e}) - f_l(\mathbf{e}'))/y_l)_l\|_1 \\
&= \|(((e_l^* - \mathbf{e}^T \mathbf{\Lambda}_{.l})^+ - (e_l^* - \mathbf{e}'^T \mathbf{\Lambda}_{.l})^+)/y_l)_l\|_1 \\
&\leq \|(((e_l^* - \mathbf{e}^T \mathbf{\Lambda}_{.l}) - (e_l^* - \mathbf{e}'^T \mathbf{\Lambda}_{.l}))/y_l)_l\|_1, \text{ since } (.)^+ \text{ is 1-lipschitz} \\
&= \|(((\mathbf{e} - \mathbf{e}')^T \mathbf{\Lambda}_{.l})/y_l)_l\|_1
\end{aligned}$$

$$= \sum_l |(\mathbf{e} - \mathbf{e}')^T \boldsymbol{\Lambda}_{\boldsymbol{.l}}| \tfrac{1}{y_l}$$

$$= \sum_l | \sum_k (e_k - e'_k) \lambda_{kl} | \tfrac{1}{y_l}$$

$$\leq \sum_l \sum_k |(e_k - e'_k)| \lambda_{kl} \tfrac{1}{y_l}$$

$$= \sum_k \sum_l |(e_k - e'_k)| \lambda_{kl} \tfrac{1}{y_l}$$

$$= \sum_k |(e_k - e'_k) \tfrac{1}{y_k}| \sum_l \lambda_{kl} \tfrac{y_k}{y_l}$$

$$\leq \sum_k |(e_k - e'_k)| \tfrac{1}{y_k} \ \Lambda^+_{\mathbf{y}}$$

$$= \|(\mathbf{e} - \mathbf{e}')\|_{1,\mathbf{y}} \Lambda^+_{\mathbf{y}}$$

where

$$\Lambda^+_{\mathbf{y}} = \max_k \sum_l \lambda_{kl} \frac{y_k}{y_l}.$$

By assumption $\Lambda^+_{\mathbf{y}} < 1$. Thus, f is a contraction with respect to $\|.\|_{1,\mathbf{y}}$, and therefore admits a unique fixed point, so $G(\boldsymbol{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium. $\square$

Proof of Theorem 2. For $\mathbf{e} \in \prod_l [0, e^*_l]$, the best response of each player l is given by

$$f_l(\mathbf{e}) = (e^*_l - \mathbf{e}^T \boldsymbol{\Lambda}_{\boldsymbol{.l}})^+.$$

Since $\boldsymbol{\Lambda} \geq 0$, $f_l(\mathbf{e}) \leq e^*_l$. Hence we can introduce the best response mapping:

$$f : \prod_l [0, e^*_l] \longrightarrow \prod_l [0, e^*_l] : \mathbf{e} \mapsto f(\mathbf{e}) = (f_l(\mathbf{e}))_l.$$

We shall establish that this mapping is a contraction with respect to the specific norm, $\| \,.\, \|_{\infty,\mathbf{y}}$. Since $\|.\|_{\infty,\mathbf{y}}$ and $\|.\|_\infty$ are equivalent norms, $\prod_l [0, e_l^*]$ remains a complete space. Let $\mathbf{e}, \mathbf{e}' \in \prod_l [0, e_l^*]$. We have:

$$
\begin{aligned}
\|f(\mathbf{e}) - f(\mathbf{e}')\|_{\infty,\mathbf{y}} &= \|(f(\mathbf{e}) - f(\mathbf{e}'))/\mathbf{y}\|_\infty \\
&= \|((\mathbf{e}^* - \mathbf{e}^T\boldsymbol{\Lambda})^+ - (\mathbf{e}^* - \mathbf{e}'^T\boldsymbol{\Lambda})^+)/\mathbf{y}\|_\infty \\
&= \max_l |(e_l^* - \mathbf{e}^T\boldsymbol{\Lambda}_{.l})^+ - (e_l^* - \mathbf{e}'^T\boldsymbol{\Lambda}_{.l})^+|\tfrac{1}{y_l} \\
&\leq \max_l |(e_l^* - \mathbf{e}^T\boldsymbol{\Lambda}_{.l}) - (e_l^* - \mathbf{e}'^T\boldsymbol{\Lambda}_{.l})|\tfrac{1}{y_l}, \text{ since } (.)^+ \text{ is 1-lipschitz} \\
&= \max_l |(\mathbf{e} - \mathbf{e}')^T\boldsymbol{\Lambda}_{.l}|\tfrac{1}{y_l} \\
&= \max_l |\textstyle\sum_k (e_k - e_k')\lambda_{kl}|\tfrac{1}{y_l} \\
&\leq \max_l \textstyle\sum_k |(e_k - e_k')|\lambda_{kl}\tfrac{1}{y_l} \\
&= \max_l \textstyle\sum_k |(e_k - e_k')|\tfrac{1}{y_k}\lambda_{kl}\tfrac{y_k}{y_l} \\
&\leq \max_l \|(\mathbf{e} - \mathbf{e}')/\mathbf{y}\|_\infty \textstyle\sum_k \lambda_{kl}\tfrac{y_k}{y_l} \\
&= \|(\mathbf{e} - \mathbf{e}')/\mathbf{y}\|_\infty \max_l \textstyle\sum_k \lambda_{kl}\tfrac{y_k}{y_l} \\
&= \|(\mathbf{e} - \mathbf{e}')\|_{\infty,\mathbf{y}}\Lambda_{\mathbf{y}}^-
\end{aligned}
$$

where

$$\Lambda_{\mathbf{y}}^- = \max_l \sum_k \lambda_{kl}\frac{y_k}{y_l}.$$

By assumption $\Lambda^-_{\mathbf{y}} < 1$. Thus, f is a contraction with respect to $\|.\|_{\infty,\mathbf{y}}$, and therefore admits a unique fixed point, so $G(\mathbf{\Lambda}, \mathbf{e}^*)$ admits a unique Nash equilibrium. □

Note that if one starts with $f_l(\mathbf{e}) = (e^*_l - \mathbf{e}^T\mathbf{\Lambda}_{.l})$, the existence of a unique solution in $\mathbb{R}^N$ to

$$\hat{\mathbf{e}} = \mathbf{e}^* - \hat{\mathbf{e}}^T\mathbf{\Lambda}$$

can be established following the last development of the proofs of Theorems 1 and 2. One may refer to Theorem 7 in Banach (1922) for more details on general linear systems of equalities. The fact that $(.)^+$ is 1-lipschitz is useful essentially for the uniqueness of Nash equilibria.

Proof of Property 2. Let $G(\mathbf{\Lambda}, \mathbf{e}^*)$ be a network game. Since $\Lambda^-_{\mathbf{e}^*} < 1$, that is $\mathbf{e}^{*T}(\boldsymbol{I} - \mathbf{\Lambda}) >> 0$, there is a unique Nash equilibrium profile $\hat{\mathbf{e}}$. Then, for every l,

$$\hat{e}_l = \max\{0, e^*_l - \hat{\mathbf{e}}^T\mathbf{\Lambda}_{.l}\} \geq \max\{0, e^*_l - \mathbf{e}^{*T}\mathbf{\Lambda}_{.l}\} > 0,$$

so $\hat{\mathbf{e}}$ is a Nash equilibrium without free riders, i.e., $\min_i \hat{e}_i > 0$. □

Proof of Property 3.

$$\Lambda^+_{\mathbf{y}} < 1 \iff \forall k,\ \sum_l \lambda_{kl}\frac{y_k}{y_l} < 1 \iff \forall k,\ \sum_l \lambda_{kl}\frac{1}{y_l} < \frac{1}{y_k}$$

$$\iff \forall k,\ \mathbf{\Lambda}_{\boldsymbol{k}.}\mathbf{1}/\mathbf{y} < \frac{1}{y_k} \iff \mathbf{1}/\mathbf{y} - \mathbf{\Lambda}.\mathbf{1}/\mathbf{y} >> 0.$$

□

Proof of Property 4.

$$\Lambda^-_{\mathbf{y}} < 1 \iff \forall l,\ \sum_k \lambda_{kl}\frac{y_k}{y_l} < 1 \iff \forall l,\ \sum_k \lambda_{kl}y_k < y_l$$

$$\iff \forall l,\ \mathbf{y}^T\mathbf{\Lambda}_{.l} < y_l \iff \mathbf{y}^T(\mathbf{I}-\mathbf{\Lambda}) >> 0 \iff \mathbf{y}-\mathbf{\Lambda}^T\mathbf{y} >> 0.$$

$\square$

Proof of Theorem 3. Let $G(\mathbf{\Lambda}, \mathbf{e}^*)$ be a network game with $\mathbf{\Lambda}$ a productive matrix. So $\mathbf{\Lambda}^T$ is productive and there exists $\mathbf{y} >> 0$ such that $\mathbf{y} >> \mathbf{\Lambda}^T\mathbf{y}$. We show that $(\mathbf{\Lambda}^T)^2$ is a productive matrix. We have,

$$\begin{aligned} \mathbf{y}-(\mathbf{\Lambda}^T)^2\mathbf{y} &>> \mathbf{\Lambda}^T\mathbf{y}-(\mathbf{\Lambda}^T)^2\mathbf{y} \\ &= \mathbf{\Lambda}^T(\mathbf{y}-\mathbf{\Lambda}^T\mathbf{y}) \\ &\geq 0, \end{aligned}$$

as $\mathbf{\Lambda}^T \geq 0$ and $\mathbf{y}-\mathbf{\Lambda}^T\mathbf{y} >> 0$. So $(\mathbf{\Lambda}^T)^2$ is a productive matrix.

We now show that $(\mathbf{I}+\mathbf{\Lambda}^T)$ is invertible. Indeed, the spectral radius of $\mathbf{\Lambda}^T$ is smaller than 1 since $\mathbf{\Lambda}$ is productive. Thus, (-1) does not belong to the spectrum of $\mathbf{\Lambda}^T$, hence 0 does not belong to the spectrum of $(\mathbf{I}+\mathbf{\Lambda}^T)$, so $(\mathbf{I}+\mathbf{\Lambda}^T)$ is invertible.

Let $\hat{\mathbf{e}} >> 0$ be the unique Nash equilibrium of the game $G(\mathbf{\Lambda}, \mathbf{e}^*)$. We have, for all l,

$$\hat{e}_l = \max\{0, e_l^* - \hat{\mathbf{e}}^T\mathbf{\Lambda}_{.l}\} > 0.$$

Thus,

$$\hat{\mathbf{e}} + \mathbf{\Lambda}^T\hat{\mathbf{e}} = \mathbf{e}^*,$$

and it follows:

$$(\mathbf{I}+\mathbf{\Lambda}^T)\hat{\mathbf{e}} = \mathbf{e}^* \iff \hat{\mathbf{e}} = (\mathbf{I}+\mathbf{\Lambda}^T)^{-1}\mathbf{e}^*$$

$$\iff \hat{\mathbf{e}} = (\mathbf{I}+\mathbf{\Lambda}^T)^{-1}(\mathbf{I}-\mathbf{\Lambda}^T)(\mathbf{I}-\mathbf{\Lambda}^T)^{-1}\mathbf{e}^*.$$

One may note that $(\mathbf{I} + \mathbf{\Lambda}^T)^{-1}$ and $(\mathbf{I} - \mathbf{\Lambda}^T)$ commute. Indeed,

$$(\mathbf{I} + \mathbf{\Lambda}^T)^{-1}(\mathbf{I} - \mathbf{\Lambda}^T) = (\mathbf{I} - \mathbf{\Lambda}^T)(\mathbf{I} + \mathbf{\Lambda}^T)^{-1}$$

is equivalent, by multiplication on the left and on the right by $(\mathbf{I} + \mathbf{\Lambda}^T)$ to

$$(\mathbf{I} - \mathbf{\Lambda}^T)(\mathbf{I} + \mathbf{\Lambda}^T) = (\mathbf{I} + \mathbf{\Lambda}^T)(\mathbf{I} - \mathbf{\Lambda}^T),$$

that is

$$\mathbf{I} - (\mathbf{\Lambda}^T)^2 = \mathbf{I} - (\mathbf{\Lambda}^T)^2.$$

So,

$$\begin{aligned}
\hat{\mathbf{e}} &= (\mathbf{I} - \mathbf{\Lambda}^T)(\mathbf{I} + \mathbf{\Lambda}^T)^{-1}(\mathbf{I} - \mathbf{\Lambda}^T)^{-1}\mathbf{e}^* \\
&= (\mathbf{I} - \mathbf{\Lambda}^T)((\mathbf{I} - \mathbf{\Lambda}^T)(\mathbf{I} + \mathbf{\Lambda}^T))^{-1}\mathbf{e}^* \\
&= (\mathbf{I} - \mathbf{\Lambda}^T)(\mathbf{I} - (\mathbf{\Lambda}^T)^2)^{-1}\mathbf{e}^* \\
&= (\mathbf{I} - \mathbf{\Lambda}^T)\mathbf{b}_{\mathbf{e}^*}(\mathbf{\Omega})
\end{aligned}$$

where $\mathbf{\Omega} = (\mathbf{\Lambda}^T)^2$. □

References

[1] BALLESTER, C., and A. CALVO-ARMENGOL (2010) Interactions with hidden complementarities, *Regional Science and Urban Economics* **40**, 397-406.

[2] BALLESTER, C., A. CALVO-ARMENGOL, and Y. ZENOU (2006) Who's who in networks. Wanted: The key player, *Econometrica* **74**, 1403-1417.

[3] BANACH, S. (1922) Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae* **3**, 133-181.

[4] BLOCH, F., and U. ZENGINOBUZ (2007) The effects of spillovers on the provision of local public goods, *Review of Economic Design* **11**, 199-216.

[5] BONACICH, P. (1987) Power and centrality: A family of measures, *American Journal of Sociology* **92**, 1170-1182.

[6] BRAMOULLE, Y., and R. KRANTON (2007) Public goods in networks, *Journal of Economic Theory* **135**, 478-494.

[7] BRAMOULLE, Y., R. KRANTON, and M. D'AMOURS (2011) Strategic interactions and networks, mimeo.

[8] DORFMAN, R., P. A. SAMUELSON, and R. M. SOLOW (1958) *Linear Programming and Economic Analysis*, Dover Publications: New York.

[9] GALE, D. (1960) *The Theory of Linear Economic Models*, McGraw-Hill: New York.

[10] ILKILIC, R. (2011) Networks of common property resources, *Economic Theory* **47**, 105-134.

[11] IVANOV, G. I. (2001) On a class of nonnegative matrices in mathematical economics models, *Comptes Rendus de l'Académie Bulgare des Sciences* **54**, 12-19.